# Indefinite Stochastic Linear Quadratic Control with Markovian Jumps in Infinite Time Horizon [*]

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**Abstract.** This paper studies a stochastic linear quadric (LQ) control problem in the infinite time horizon with Markovian jumps in parameter values. In contrast to the deterministic case, the cost weighting matrices of the state and control are allowed to be indinfite here. When the generator matrix of the jump process – which is assumed to be a Markov chain – is known and time-invariant, the well-posedness of the indefinite stochastic LQ problem is shown to be equivalent to the solvability of a system of coupled generalized algebraic Riccati equations (CGAREs) that involves equality and inequality constraints. To analyze the CGAREs, linear matrix inequalities (LMIs) are utilized, and the equivalence between the feasibility of the LMIs and the solvability of the CGAREs is established. Finally, an LMI-based algorithm is devised to slove the CGAREs via a semidefinite programming, and numerical results are presented to illustrate the proposed algorithm.

**Key words:** Stochastic LQ control, coupled generalized algebraic Riccati equations, linear matrix inequality, semidefinite programming, mean-square stability.

## 1. Introduction

In this paper, we consider indefinite stochastic linear quadratic (LQ) control with jumps in the following form:

$$\min \quad E\left\{\int_0^{+\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \,\middle|\, r_0 = i\right\},$$

$$\text{subject to} \quad \begin{cases} dx(t) = [A(r_t)x(t) + B(r_t)u(t)]dt + [C(r_t)x(t) + D(r_t)u(t)]dW(t), \\ x(0) = x_0 \in \mathbf{R}^n, \end{cases}$$

where $r_t$ is a Markov chain taking values in $\{1, \cdots, l\}$, $W(t)$ is a Brownian motion independent of $r_t$, and $A(r_t) = A_i$, $B(r_t) = B_i$, $C(r_t) = C_i$, $D(r_t) = D_i$ $Q(r_t) = Q_i$, $R(r_t) = R_i$ and $L(r_t) = L_i$ when $r_t = i$ ($i = 1, \cdots, l$). Here the matrices $A_i$, etc. are given with appropriate dimensions. The Markov chain $r_t$ has the transition probabilities given by:

$$\mathbf{P}\{r_{t+\Delta t} = j | r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{else,} \end{cases} \tag{1}$$

where $\pi_{ij} \geqslant 0$ for $i \neq j$ and $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$.

The stochastic LQ control problem, initiated by Wonham [23], is one of the most fundamental tools in modern engineering. In most literature, it is a common assumption that the cost weighting matrix of control be positive definite (see [6, 9]). However, this assumption has been challenged by some recent works ([2, 3, 8]) that a class of stochastic LQ problems with indefinite control weights may still be sensible and well-posed. Note that this phenomenon may occur only when the diffusion coefficient of the system dynamics depends on the control, meaning that controls could or would influence the uncertainty scale in the system.

On the other hand, studies on the stochastic model of jump linear systems can be traced back at least to the work of Krasosvkii and Lidskii [14]. During the last decade, LQ control problems with jumps have been extensively studied; see, for example, Ait Rami and El Ghaoui [1], Mariton [15], Ji and Chizeck [12, 13], and Zhang and Yin [25]. However, the existing works usually set the diffusion coefficients as either 0 or $\sigma(r_t)(\neq 0)$ independent of the state or control. As a result, they have to assume, again, that the control cost weighting matrices be positive definite. To elaborate, take the special case when the diffusion term is absent. Assuming that the state weighting matrix in the cost is non-negative definite and the control weighting matrix is positive definite, the LQ control problem is automatically well-posed and can be solved via the system of coupled algebraic Riccati equations (CAREs).

$$A_i' P_i + P_i A_i - (P_i B_i + L_i) R_i^{-1} (P_i B_i + L_i)' + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j = 0, \quad (2)$$
$$i = 1, \cdots, l.$$

Moreover, it can be shown that this system has a solution $(P_1^*, \cdots, P_l^*)$ based on which the optimal control is represented as

$$u^*(t) = -\sum_{i=1}^{l} R_i^{-1} (P_i^* B_i + L_i)' x^*(t) \chi_{\{r_t=i\}}(t), \quad (3)$$

where $\chi(t)$ is the indicator function. However, in many real problems the analytical solutions to the CAREs (2) are very hard to obtain. Several numerical algorithms have been therefore proposed for solving the coupled Riccati equations; see, e.g., Wonham [24] and Mariton and Bertrand [16]. Recently, an algorithm based on convex optimization over linear matrix inequalities (LMIs):

$$\left[ \begin{array}{c|c} A_i' P_i + P_i A_i + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j & P_i B_i + L_i \\ \hline B_i' P_i + L_i' & R_i \end{array} \right] \geqslant 0, \quad i = 1, \cdots, l, \quad (4)$$

put forward by Ait Rami and El Ghaoui [1], successfully solves the CAREs (2) in polynomial time, using currently available software [11].

Now, if we extend the above special case to the indefinite LQ case to be studied in this paper, we must consider the following system of coupled generalized

algebraic Riccati equations (CGAREs):

$$\begin{cases} A_i'P_i + P_iA_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l} \pi_{ij}P_j \\ \quad -(P_iB_i+C_i'P_iD_i+L_i)(R_i+D_i'P_iD_i)^{-1}(P_iB_i+C_i'P_iD_i+L_i)'=0, \quad (5) \\ R_i + D_i'P_iD_i > 0, \qquad\qquad\qquad i = 1, \cdots, l. \end{cases}$$

If there exists a solution $(P_1^*, \cdots, P_l^*)$ to the above equation with $R_i + D_i'P_i^*D_i > 0$ $(i = 1, \cdots, l)$, then a possible optimal feedback control would be

$$u^*(t) = -\sum_{i=1}^{l}(R_i + D_i'P_i^*D_i)^{-1}(P_i^*B_i + C_i'P_i^*D_i + L_i)'x^*(t)\chi_{\{r_t=i\}}(t). \quad (6)$$

However, there are some fundamental differences and difficulties with the CGAREs (5) compared to its special case CAREs (2). First, the equality constraint part of the CGAREs (5) is more complicated than its counterpart in CAREs (2) for the inverses now involve the unknown $(P_1, \cdots, P_l)$. Second, there exist $l$ *additional* strictly positive definiteness constraints in the equations.

In this paper, we develop an analytical and computational approach to solving the CGAREs (5). The key idea is to utilize LMIs of the following form

$$\begin{cases} \left[ \begin{array}{c|c} A_i'P_i + P_iA_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l} \pi_{ij}P_j & P_iB_i + C_i'P_iD_i + L_i \\ \hline B_i'P_i + D_i'P_iC_i + L_i' & R_i + D_i'P_iD_i \end{array} \right] \geqslant 0, \\ R_i + D_i'P_iD_i > 0, \qquad\qquad\qquad i = 1, \cdots, l \end{cases} \quad (7)$$

as a powerful tool. A consequence of this formulation is that the problem can be conveniently solved in polynomial time based on solving a semidefinite programming (SDP) [7, 22]. Moreover, we show that, provided that the systems is stabilizing in the mean-square sense, our approach always yields the *maximal* solution to the CGAREs (5), which in turn guarantees that (6) is indeed an optimal feedback control.

The remainder of the paper is organized as follows. In Section 2 we formulate the indefinite stochastic LQ problem with jumps in infinite time horizon and present some preliminaries. In Section 3 some relations between the well-posedness of the LQ problem and the feasibility of the corresponding LMIs are established. In Section 4, we further show that the feasibility of the LMIs is equivalent to the solvability of the CGAREs, and present an algorithm of obtaining the maximal solution of the CGAREs via an SDP. Section 5 characterizes the optimal control to the original LQ control problem in terms of the maximal solution to the CGAREs (5). Section 6 presents some illustrative numerical examples and Section 7 gives some concluding remarks. The proofs of all the theorems are supplied in an Appendix.

## 2. Problem Formulation and Preliminaries

### 2.1. NOTATION

We make use of the following basic notation in this paper:

| | | |
|---|---|---|
| $\mathbf{R}^n$ | : | $n$-dimensional Euclidean space; |
| $\mathbf{R}^{n\times m}$ | : | the set of all $n \times m$ matrices; |
| $\mathscr{S}^n$ | : | the set of all $n \times n$ symmetric matrices; |
| $\mathscr{S}^n_+$ | : | the subset of all nonnegative definite matrices of $\mathscr{S}^n$; |
| $\hat{\mathscr{S}}^n_+$ | : | the subset of all positive definite matrices of $\mathscr{S}^n$; |
| $(\mathscr{S}^n)^l$ | : | $= \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{l}$; |
| $(\mathscr{S}^n_+)^l$ | : | $= \underbrace{\mathscr{S}^n_+ \times \cdots \times \mathscr{S}^n_+}_{l}$; |
| $(\hat{\mathscr{S}}^n_+)^l$ | : | $= \underbrace{\hat{\mathscr{S}}^n_+ \times \cdots \times \hat{\mathscr{S}}^n_+}_{l}$; |
| $M'$ | : | the transpose of any matrix $M$; |
| $M > 0$ | : | the symmetric matrix $M$ is positive definite; |
| $M \geqslant 0$ | : | the symmetric matrix $M$ is nonnegative definite; |
| $\mathbf{Tr}(M)$ | : | the trace of any square matrix $M$; |
| $|M|$ | : | $= \sqrt{\mathbf{Tr}(MM')}$; |
| $\chi_A$ | : | the indicator function of a set $A$; |
| $L^\infty(0, T; \mathbf{R}^{n\times m})$ | : | the set of essentially bounded measurable functions $\phi : [0, T] \to \mathbf{R}^{n\times m}$. |

### 2.2. PROBLEM FORMULATION

First of all, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ be a given filtered probability space where there live a standard one-dimensional Brownian motion $W(t)$ on $[0, +\infty)$ (with $W(0) = 0$) and a Markov chain $r_t \in \{1, 2, \cdots, l\}$ with the generator $\Pi = (\pi_{ij})$, and $\mathcal{F}_t = \sigma\{W(s), r_s | 0 \leqslant s \leqslant t\}$. The Brownian motion is assumed to be one dimensional only for simplicity; there is no essential difference for the multi-dimensional case. In addition, the process $r_t$ and $W(t)$ are assumed to be independent throughout this paper. Define

$$L_2^{loc}(\mathbf{R}^{nu}) = \left\{ \begin{array}{c} \phi(\cdot, \cdot) : \\ {[0, +\infty)} \times \Omega \mapsto \mathbf{R}^{nu} \end{array} \middle| \begin{array}{l} \phi(\cdot, \cdot) \text{ is } \mathcal{F}_t\text{-adapted, Lebesgue measurable,} \\ \text{and } E \int_0^T |\phi(t, \omega)|^2 dt < +\infty, \ \forall T \geqslant 0 \end{array} \right\}.$$

Consider the linear stochastic differential equation subject to Markovian jumps defined by

$$\begin{cases} dx(t) = [A(r_t)x(t) + B(r_t)u(t)]dt + [C(r_t)x(t) + D(r_t)u(t)]dW(t) \\ x(0) = x_0 \in \mathbf{R}^n, \end{cases} \quad (8)$$

where $A(r_t) = A_i$, $B(r_t) = B_i$, $C(r_t) = C_i$ and $D(r_t) = D_i$ when $r_t = i$, while $A_i$, etc., $i = 1, 2, \cdots, l$, are given matrices of suitable sizes. A process $u(\cdot)$ is called a control if $u(\cdot) \in L_2^{loc}(\mathbf{R}^{nu})$.

DEFINITION 2.1. A control $u(\cdot)$ is called (mean-square) stabilizing with respect to (w.r.t.) a given initial state $(x_0, i)$ if the corresponding state $x(\cdot)$ of (8) with $x(0) = x_0$ and $r_0 = i$ satisfies $\lim\limits_{t \to +\infty} E[x(t)'x(t)] = 0$.

DEFINITION 2.2. The system (8) is called (mean-square) stabilizable if there exists a feedback control $u(t) = \sum_{i=1}^{l} K_i x(t) \chi_{\{r_t=i\}}(t)$, where $K_1, \cdots, K_l$ are given matrices, which is stabilizing w.r.t. any initial state $(x_0, i)$.

Next, for a given $(x_0, i) \in \mathbf{R}^n \times \{1, 2, \cdots, l\}$, we define the corresponding set of admissible controls:

$$\mathcal{U}(x_0, i) \triangleq \left\{ u(\cdot) \in L_2^{loc}(\mathbf{R}^{n_u}) \mid u(\cdot) \text{ is mean-square stabilizing w.r.t. } (x_0, i) \right\},$$

where the integer $n_u$ is the dimension of the control variable. It is easily seen that $\mathcal{U}(x_0, i)$ is a convex subset of $L_2^{loc}(\mathbf{R}^{n_u})$.

For each $(x_0, i, u(\cdot)) \in \mathbf{R}^n \times \{1, 2, \cdots, l\} \times \mathcal{U}(x_0, i)$, the optimal control problem is to find a control which minimizes the following quadratic cost associated with (8)

$$J(x_0, i; u(\cdot)) = E\left\{ \int_0^{+\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \,\middle|\, r_0 = i \right\}, \quad (9)$$

where $Q(r_t) = Q_i$, $R(r_t) = R_i$ and $L(r_t) = L_i$ when $r_t = i$, while $Q_i$, etc., $i = 1, 2, \cdots, l$, are given matrices with suitable sizes. The value function $V$ is defined as

$$V(x_0, i) = \inf_{u(\cdot) \in \mathcal{U}(x_0, i)} J(x_0, i; u(\cdot)). \quad (10)$$

Since the symmetric matrices

$$\begin{bmatrix} Q_i & L_i \\ L_i' & R_i \end{bmatrix} i = 1, \cdots, l,$$

are allowed to be indefinite, the above optimization problem is referred to as an indefinite LQ problem. It should be noted that due to the indefiniteness the cost functional $J(x_0, i; u(\cdot))$ is not necessarily convex in $u(\cdot)$.

DEFINITION 2.3. The LQ problem is called well-posed if

$$-\infty < V(x_0, i) < +\infty, \quad \forall x_0 \in \mathbf{R}^n, \quad \forall i = 1, \cdots, l. \tag{11}$$

A well-posed problem is called attainable (w.r.t. $(x_0, i)$) if there is a control $u^*(\cdot) \in \mathcal{U}(x_0, i)$ that achieves $V(x_0, i)$. In this case the control $u^*(\cdot)$ is called optimal (w.r.t. $(x_0, i)$).

The following two basic assumptions are imposed throughout this paper.

ASSUMPTION 2.1. The system (8) is mean-square stabilizable.

Mean-square stabilizability is a standard assumption in an infinite-horizon LQ control problem. In words, it basically ensures that there is at least one meaningful control, in the sense that the corresponding state trajectory is square integrable (hence does not "blow up"), with respect to any initial conditions. The problem would be trivial without this assumption.

ASSUMPTION 2.2. The data appearing in the LQ problem $(8) - (9)$ satisfy, for every $i$,

$$A_i, C_i \in \mathbf{R}^{n \times n}, \ B_i, D_i \in \mathbf{R}^{n \times n_u}, \ Q_i \in \mathcal{S}^n, \ L_i \in \mathbf{R}^{n \times n_u}, \ R_i \in \mathcal{S}^{n_u}.$$

## 2.3. SOME LEMMAS

In this subsection we list some lemmas that are important in our subsequent analysis. First of all, we present a generalized Itô's formula featuring diffusion processes with jumps.

LEMMA 2.1. (Generalized Itô's formula [5]). *Let $b(t, \omega, i)$ and $\sigma(t, \omega, i)$ be given $\mathbf{R}^n$-valued, $\mathcal{F}_t$-adapted processes, $i = 1, 2, \cdots, l$, and*

$$dx(t) = b(t, \omega, r_t)dt + \sigma(t, \omega, r_t)dW(t).$$

*Then for given $\varphi(\cdot, \cdot, i) \in C^2([0, \infty) \times \mathbf{R}^n)$, $i = 1, \cdots, l$, we have*

$$E\left\{\varphi(T, x(T), r_T) - \varphi(s, x(s), r_s) \mid r_s = i\right\} = E\left\{\int_s^T \Gamma\varphi(t, x(t), r_t)dt \ \Big| \ r_s = i\right\}, \tag{12}$$

*where*

$$\begin{aligned}\Gamma\varphi(t, x, i) &= \varphi_t(t, x, i) + b(t, \omega, i)'\varphi_x(t, x, i) \\ &\quad + \tfrac{1}{2}\mathrm{tr}[\sigma(t, \omega, i)'\varphi_{xx}(t, x, i)\sigma(t, \omega, i)] + \sum_{j=1}^{l} \pi_{ij}\varphi(t, x, j).\end{aligned}$$

Next, we recall some of the basic properties of the pseudo inverse of a matrix.

LEMMA 2.2. ([19]). *Let a matrix $M \in \mathbf{R}^{m \times n}$ be given. Then there exists a unique matrix $M^\dagger \in \mathbf{R}^{n \times m}$ such that*

$$MM^\dagger M = M, \ M^\dagger MM^\dagger = M^\dagger, \ (MM^\dagger)' = MM^\dagger, \ (M^\dagger M)' = M^\dagger M. \quad (13)$$

The matrix $M^\dagger$ above is called the Moore-Penrose pseudo inverse of $M$.

LEMMA 2.3. ([3]). *For a symmetric matrix S, we have*
(i)     $S^\dagger = (S^\dagger)'$;
(ii)    $SS^\dagger = S^\dagger S$;
(iii)   $S \geqslant 0$ *if and only if* $S^\dagger \geqslant 0$.

   Finally, the following generalized version of the well-known Schur lemma [4] involving pseudo inverse plays a key technical role in this paper.

LEMMA 2.4. (Extended Schur's lemma [3]). *Let matrices $M = M'$, $N$ and $R = R'$ be given with appropriate dimensions. Then the following conditions are equivalent:*
(i)     $M - NR^\dagger N' \geqslant 0$ *and* $N(I - RR^\dagger) = 0$, $R \geqslant 0$;

(ii)    $\begin{bmatrix} M & N \\ N' & R \end{bmatrix} \geqslant 0$;

(iii)   $\begin{bmatrix} R & N' \\ N & M \end{bmatrix} \geqslant 0$.

## 2.4. CGAREs AND LMIs

Define a subset $\mathit{l}$ of $(\mathscr{S}^n)^l$:

$$\mathit{l} \triangleq \left\{ (X_1, \cdots, X_l) \in (\mathscr{S}^n)^l \ \big| \ \mathbf{Det}(R_i + D_i' X_i D_i) \neq 0, \ i = 1, \cdots, l \right\}. \quad (14)$$

Assume that $\mathit{l} \neq \emptyset$, which is satisfied when, say, $\mathbf{Ker}R_i \cap \mathbf{Ker}D_i \neq \emptyset$, $\forall i = 1, \cdots, l$. Define the operator $\mathcal{R}_i : \mathit{l} \to \mathscr{S}^n$ by

$$\begin{aligned}
&\mathcal{R}_i(X_1, \cdots, X_l) \\
&\triangleq A_i' X_i + X_i A_i + C_i' X_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} X_j \\
&\quad - (X_i B_i + C_i' X_i D_i + L_i)(R_i + D_i' X_i D_i)^{-1}(B_i' X_i + D_i' X_i C_i + L_i'), \\
&i = 1, \cdots, l.
\end{aligned} \quad (15)$$

   Associated with the stochastic LQ problem (8)–(9) there is a system of CGAREs:

$$\begin{cases} \mathcal{R}_i(P_1, \cdots, P_l) = 0, \\ R_i + D_i' P_i D_i > 0, \qquad i = 1, \cdots, l. \end{cases} \quad (16)$$

   The key idea of this paper is to reformulate the CGAREs as LMIs, which is a powerful tool to treat the original LQ problem by using convex optimization techniques. Let us first introduce the general notion of LMIs [22].

DEFINITION 2.4. Let symmetric matrices $F_0, F_1, \cdots, F_m \in \mathcal{S}^n$ be given. Inequalities consisting of any combination of the following relations

$$\begin{cases} F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i > 0, \\ F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i \geqslant 0, \end{cases} \tag{17}$$

are called LMIs with respect to the variable $x = (x_1, \dots, x_m)' \in \mathbf{R}^m$.

The LMIs associated with the CGAREs (16) are

$$\begin{cases} \left[ \begin{array}{c|c} A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j & P_i B_i + C_i' P_i D_i + L_i \\ \hline B_i' P_i + D_i' P_i C_i + L_i' & R_i + D_i' P_i D_i \end{array} \right] \geqslant 0, \\ R_i + D_i' P_i D_i > 0, \qquad\qquad\qquad i = 1, \cdots, l, \end{cases} \tag{18}$$

with respect to the variables $(P_1, \cdots, P_l) \in (\mathcal{S}^n)^l$.


## 2.5. MEAN-SQUARE STABILIZABILITY

Mean-square stabilizability is an important issue that needs to be addressed for the LQ problem in the infinite time horizon. The following lemma, originally proved in [10], relates the stabilizability of the system (8) to the feasibility of certain coupled Lyapunov inequalities that are essentially LMIs.

LEMMA 2.5. ([10]). *The following properties are equivalent.*
(i)     *System* (8) *is mean-square stabilizable.*
(ii)    *There exist matrices $K_1, \cdots, K_l$ and symmetric matrices $X_1, \cdots, X_l$ such that*
$$\begin{cases} (A_i + B_i K_i)' X_i + X_i (A_i + B_i K_i) + (C_i + D_i K_i)' X_i (C_i + D_i K_i) \\ \quad + \sum_{j=1}^l \pi_{ij} X_j < 0, \\ X_i > 0, \qquad\qquad\qquad i = 1, \cdots, l. \end{cases} \tag{19}$$
*In this case the feedback $u(t) = \sum_{i=1}^l K_i x(t) \chi_{\{r_t=i\}}(t)$ is stabilizing w.r.t. any initial $(x_0, i)$.*
(iii)   *There exist matrices $K_1, \cdots, K_l$ and symmetric matrices $X_1, \cdots, X_l$ such that*
$$\begin{cases} (A_i + B_i K_i) X_i + X_i (A_i + B_i K_i)' + (C_i + D_i K_i) X_i (C_i + D_i K_i)' \\ \quad + \sum_{j=1}^l \pi_{ji} X_j < 0, \\ X_i > 0, \qquad\qquad\qquad i = 1, \cdots, l. \end{cases} \tag{20}$$
*In this case the feedback $u(t) = \sum_{i=1}^l K_i x(t) \chi_{\{r_t=i\}}(t)$ is stabilizing w.r.t. any initial $(x_0, i)$.*
(iv)    *There exist matrices $K_1, \cdots, K_l$ such that, for all matrices $Y_1, \cdots, Y_l$, there exists a unique solution $(X_1, \cdots, X_l)$ to the following matrix equations*
$$\begin{aligned} (A_i + B_i K_i)' X_i + X_i (A_i + B_i K_i) + (C_i + D_i K_i)' X_i (C_i + D_i K_i) \\ + \sum_{j=1}^l \pi_{ij} X_j + Y_i = 0, \quad i = 1, \cdots, l. \end{aligned} \tag{21}$$

    *If, for every $i$, $Y_i > 0$ (resp. $Y_i \geqslant 0$) then $X_i > 0$ (resp. $X_i \geqslant 0$). Furthermore, in this case the feedback $u(t) = \sum_{i=1}^{l} K_i x(t) \chi_{\{r_t=i\}}(t)$ is stabilizing w.r.t. any initial $(x_0, i)$.*

(v)   *There exist matrices $K_1, \cdots, K_l$ such that, for all matrices $Y_1, \cdots, Y_l$, there exists a unique solution $(X_1, \cdots, X_l)$ to the following matrix equations*

$$(A_i + B_i K_i)X_i + X_i(A_i + B_i K_i)' + (C_i + D_i K_i)X_i(C_i + D_i K_i)' \\ + \sum_{j=1}^{l} \pi_{ji} X_j + Y_i = 0, \quad i = 1, \cdots, l. \tag{22}$$

    *If, for every $i$, $Y_i > 0$ (resp. $Y_i \geqslant 0$) then $X_i > 0$ (resp. $X_i \geqslant 0$). Furthermore, in this case the feedback $u(t) = \sum_{i=1}^{l} K_i x(t) \chi_{\{r_t=i\}}(t)$ is stabilizing w.r.t. any initial $(x_0, i)$.*

(vi)  *There exist matrices $Y_1, \cdots, Y_l$ and symmetric matrices $X_1, \cdots, X_l$ such that*

$$\left[ \begin{array}{c|c} A_i X_i + X_i A_i' + B_i Y_i + Y_i' B_i' + \sum_{j=1}^{l} \pi_{ij} X_j & C_i X_i + D_i Y_i \\ \hline X_i C_i' + Y_i' D_i' & -X_i \end{array} \right] < 0, \\ i = 1, \cdots, l. \tag{23}$$

    *In this case the feedback $u(t) = \sum_{i=1}^{l} Y_i X_i^{-1} x(t) \chi_{\{r_t=i\}}(t)$ is stabilizing w.r.t. any initial $(x_0, i)$.*

The above result also gives an efficient numerical way of checking mean-square stabilizability by using LMIs.

## 3. Well-posedness of LQ Problem

Before looking for an optimal control of the LQ problem, we study its well-posedness via the feasibility of the associated LMIs.

LEMMA 3.1. *Let matrices $M_1, \cdots, M_l \in \mathcal{S}^n$ be given, and $M(r_t) = M_i$ while $r_t = i$. Then for any admissible pair $(x(\cdot), u(\cdot))$ of the system (8), we have*

$$E \left\{ \int_0^T x(t)'[A(r_t)'M(r_t) + M(r_t)A(r_t) + C(r_t)'M(r_t)C(r_t) \\ + \sum_{j=1}^{l} \pi_{r_t j} M_j]x(t) + 2u(t)'[B(r_t)'M(r_t) + D(r_t)'M(r_t)C(r_t)]x(t) \\ + u(t)'D(r_t)'M(r_t)D(r_t)u(t)dt \, \Big| \, r_0 = i \right\} \tag{24}$$
$$= E \left[ x(T)'M(r_T)x(T) - x(0)'M(r_0)x(0) \mid r_0 = i \right].$$

*Proof.* Setting $\varphi(t, x, i) = x'M_i x$ and applying Lemma 2.1 to the system (8), we have

$$E \left[ x(T)'M(r_T)x(T) - x(0)'M(r_0)x(0) \mid r_0 = i \right] \\ = E \left[ \varphi(T, x(T), r_T) - \varphi(0, x(0), r_0) \mid r_0 = i \right] \\ = E \left\{ \int_0^T \Gamma\varphi(t, x(t), r_t)dt \, \Big| \, r_0 = i \right\},$$

where

$$\Gamma\varphi(t,x,i) = \varphi_t(t,x,i) + b(t,x,u,i)'\varphi_x(t,x,i)$$
$$+ \tfrac{1}{2}\mathrm{tr}[\sigma(t,x,u,i)'\varphi_{xx}(t,x,i)\sigma(t,x,u,i)] + \sum_{j=1}^{l}\pi_{ij}\varphi(t,x,j)$$
$$= x'[A_i'M_i + M_iA_i + C_i'M_iC_i + \sum_{j=1}^{l}\pi_{ij}M_j]x$$
$$+ 2u'[B_i'M_i + D_i'M_iC_i]x + u'D_i'M_iD_iu.$$

This proves the lemma.                                                               □

LEMMA 3.2. *Let the matrices $K_1, \cdots, K_l$ be specified as in Lemma 2.5 $-$ ($iv$), and $(P_1, \cdots, P_l) \in (\mathscr{S}^n)^l$ be the unique solution of the following matrix equations*

$$(A_i + B_iK_i)'P_i + P_i(A_i + B_iK_i) + (C_i + D_iK_i)'P_i(C_i + D_iK_i) \tag{25}$$
$$+ \sum_{j=1}^{l}\pi_{ij}P_j = -Q_i - L_iK_i - K_i'L_i' - K_i'R_iK_i, \qquad i = 1, \cdots, l.$$

*Then the cost corresponding to the control $u(t) = \sum_{i=1}^{l} K_ix(t)\chi_{\{r_t=i\}}(t)$ with the initial condition $(x_0, i)$ is*

$$J(x_0, i; u(\cdot)) = x_0'P_ix_0, \quad i = 1, \cdots, l. \tag{26}$$

*Proof.* Let $P(r_t) = P_i$ and $K(r_t) = K_i$ for $r_t = i$. Applying Lemma 3.1 to $M_i = P_i$ and $u(t) = \sum_{i=1}^{l} K_ix(t)\chi_{\{r_t=i\}}(t)$, we have

$$J(x_0, i; u(\cdot))$$
$$= E\left\{\int_0^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \,\Big|\, r_0 = i\right\}$$
$$= E\left\{\int_0^T x(t)'[Q(r_t) + L(r_t)K(r_t) + K(r_t)'L(r_t)' \right.$$
$$\left. + K(r_t)'R(r_t)K(r_t)]x(t)dt \,\Big|\, r_0 = i\right\}$$
$$= -E\left\{\int_0^T x(t)'[(A(r_t) + B(r_t)K(r_t))'P(r_t) + P(r_t)(A(r_t) + B(r_t)K(r_t)) \right.$$
$$+ (C(r_t) + D(r_t)K(r_t))'P(r_t)(C(r_t) + D(r_t)K(r_t))$$
$$\left. + \sum_{i=1}^{l}\pi_{r_tj}P_j]x(t)dt \,\Big|\, r_0 = i\right\}$$
$$= -E\left\{\int_0^T x(t)'[A(r_t)'P(r_t) + P(r_t)A(r_t) + C(r_t)'P(r_t)C(r_t) \right.$$
$$+ \sum_{i=1}^{l}\pi_{r_tj}P_j]x(t) + 2u(t)'[B(r_t)'P(r_t) + D(r_t)'P(r_t)C(r_t)]x(t)$$
$$\left. + u(t)'D(r_t)'P(r_t)D(r_t)u(t)dt \,\Big|\, r_0 = i\right\}$$
$$= E\left[x(0)'P(r_0)x(0) - x(T)'P(r_T)x(T) \mid r_0 = i\right]$$
$$= x_0'P_ix_0 - E[x(T)'P(r_T)x(T)].$$

Letting $T \to +\infty$, we obtain $J(x_0, i; u(\cdot)) = x_0'P_ix_0$.                  □

Next, let us define a subset $\mathcal{P}$ of $(\mathcal{S}^n)^l$:

$$\mathcal{P} \triangleq \Big\{ (P_1, \cdots, P_l) \in (\mathcal{S}^n)^l \mid \mathcal{R}_i(P_1, \cdots, P_l) \geqslant 0,$$
$$R_i + D_i' P_i D_i > 0, \ i = 1, \cdots, l \Big\}. \tag{27}$$

THEOREM 3.1. *Assume that $\mathcal{P} \neq \emptyset$. Then*

(i)      *$\mathcal{P}$ is a convex set.*

(ii)     *$\mathcal{P}$ is a bounded set in the following sense: There exist $(\tilde{P}_1, \cdots, \tilde{P}_l) \in (\mathcal{S}^n)^l$ such that $P_i \leqslant \tilde{P}_i \ (i = 1, \cdots, l), \forall (P_1, \cdots, P_l) \in \mathcal{P}$.*

THEOREM 3.2. *If $\mathcal{P} \neq \emptyset$, then the LQ problem $(8)-(9)$ is well-posed. Moreover, we have*

(i)      *$V(x_0, i) \geqslant x_0' P_i x_0, \forall x_0 \in \mathbf{R}^n, \forall i = 1, \cdots, l, \forall (P_1, \cdots, P_l) \in \mathcal{P}$.*

(ii)     *If $\mathcal{P} \cap (\mathcal{S}_+^n)^l \neq \emptyset$, then $V(x_0, i) \geqslant 0, \forall x_0 \in \mathbf{R}^n, \forall i = 1, \cdots, l$.*

The proofs of Theorem 3.1 and Theorem 3.2 can be found in Appendix. The following result is straightforward.

COROLLARY 3.1. *If the CGAREs (16) admit a solution, then the LQ problem $(8) - (9)$ is well-posed.*

## 4. Solving CGAREs via LMIs

In this section, we develop analytical and computational approach to solving the CGAREs via the LMIs and the associated SDP.

Set

$$\mathcal{G} = \{(P_1, \cdots, P_l) \in (\mathcal{S}^n)^l \mid R_i + D_i' P_i D_i > 0, \ i = 1, \cdots, l\}.$$

DEFINITION 4.1. A solution $(P_1, \cdots, P_l) \in \mathcal{G}$ of the CGAREs (16) is called its maximal solution if for any $(\tilde{P}_1, \cdots, \tilde{P}_l) \in \mathcal{G}$ with $\mathcal{R}_i(\tilde{P}_1, \cdots, \tilde{P}_l) \geqslant 0$, it holds $P_i - \tilde{P}_i \geqslant 0$, for $i = 1, \cdots, l$.

It is evident from the above definition that the maximal solution must be unique if it exists. We also show in this section that, provided that the system is mean-square stabilizable, our approach always yields the maximal solution to the CGAREs (16).

### 4.1. CGAREs VS. SDP AND ITS DUAL

First of all, let us recall some definition and results about primal SDP problems and their duals.

DEFINITION 4.2. Let a vector $c = (c_1, \cdots, c_m)' \in \mathbf{R}^m$ and matrices $F_0, F_1, \cdots,$ $F_m \in \mathcal{S}^n$ be given. The following optimization problem

$$\begin{cases} \min \quad c'x, \\ \text{s.t.} \quad F(x) \equiv F_0 + \sum_{i=1}^m x_i F_i \geqslant 0 \end{cases} \tag{28}$$

is called an SDP. Moreover, the dual problem of the SDP (28) is defined as

$$\begin{cases} \max \quad -\mathbf{Tr}(F_0 Z), \\ \text{s.t.} \quad Z \in \mathcal{S}^n, \quad \mathbf{Tr}(Z F_i) = c_i, \quad i = 1, \cdots, m, \quad Z \geqslant 0. \end{cases} \tag{29}$$

Let $p^*$ denote the infimum value of the primal SDP (28) and $d^*$ the supremum value of its dual (29). Then we have the following results (see [22]).

PROPOSITION 4.1. $p^* = d^*$ *if either of the following conditions holds:*
(i)   *The primal problem* (28) *is strictly feasible, i.e., there exists an x such that* $F(x) > 0$.
(ii)  *The dual problem* (29) *is strictly feasible, i.e., there exists a* $Z \in \mathcal{S}^n$ *with* $Z > 0$ *and* $\mathbf{Tr}(Z F_i) = c_i, \ i = 1, \cdots, m$.
*If both conditions* (i) *and* (ii) *hold, then the optimal sets of both the primal and the dual are nonempty. In this case, the following complementary slackness condition*

$$F(x)Z = 0 \tag{30}$$

*is necessary and sufficient for achieving the optimal values for both problems.*

Now we turn to consider the CGAREs

$$\begin{cases} \mathcal{R}_i(P_1, \cdots, P_l) \equiv A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j \\ \quad -(P_i B_i + C_i' P_i D_i + L_i)(R_i + D_i' P_i D_i)^{-1}(B_i' P_i + D_i' P_i C_i + L_i') = 0, \quad (31) \\ R_i + D_i P_i D_i > 0, \quad\quad\quad\quad\quad\quad i = 1, \cdots, l. \end{cases}$$

In this subsection, we pose an additional assumption that the interior of the set $\mathcal{P}$ is nonempty, namely, there exists $(P_1^0, \cdots, P_l^0) \in (\mathcal{S}^n)^l$ such that $\mathcal{R}_i(P_1^0, \cdots, P_l^0) > 0$ and $R_i + D_i' P_i^0 D_i > 0 \ (i = 1, \cdots, l)$. (In the next subsection we shall drop this assumption.) Consider the following SDP problem

$$\max \quad \sum_{i=1}^l \mathbf{Tr}(P_i)$$
$$\text{s.t.} \quad \begin{cases} \left[ \begin{array}{c|c} A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j & P_i B_i + C_i' P_i D_i + L_i \\ \hline B_i' P_i + D_i' P_i C_i + L_i' & R_i + D_i' P_i D_i \end{array} \right] \geqslant 0, \\ P_i - P_i^0 \geqslant 0, \quad\quad\quad\quad\quad\quad i = 1, \cdots, l. \end{cases}$$
$$\tag{32}$$

REMARK 4.1. For $(P_1, \cdots, P_l) \in \mathcal{P}$, we have the coupled matrices $(i = 1, \cdots, l)$:

$$\begin{aligned} &\mathcal{N}_i(P_1, \cdots, P_l) \\ &\triangleq \left[ \begin{array}{c|c} \begin{array}{c|c} A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j & P_i B_i + C_i' P_i D_i + L_i \\ \hline B_i' P_i + D_i' P_i C_i + L_i' & R_i + D_i' P_i D_i \end{array} & 0 \\ \hline 0 & P_i - P_i^0 \end{array} \right]. \end{aligned}$$
$$\tag{33}$$

The constraints of (32) can be equivalently expressed as a single LMI

$$\mathcal{N}(P_1, \cdots, P_l) \triangleq \mathbf{diag}(\mathcal{N}_1(P_1, \cdots, P_l), \cdots, \mathcal{N}_l(P_1, \cdots, P_l)) \geqslant 0. \quad (34)$$

REMARK 4.2. *The problem (32) is strictly feasible: Indeed, $(P_1^0 + \epsilon I, \cdots, P_l^0 + \epsilon I)$ is a strictly feasible solution for sufficiently small $\epsilon > 0$, where $I$ is the $n \times n$ identity matrix.*

REMARK 4.3. *For any feasible point $(P_1, \cdots, P_l)$ of (34), we have $R_i + D_i' P_i D_i > 0$, $(i = 1, \cdots, l)$. This is evident from the fact that $(P_1^0, \cdots, P_l^0)$ is a strictly feasible point of the first constraint in (32).*

THEOREM 4.1. *The dual problem of* (32) *can be formulated as follows*

$$\max \ -\sum_{i=1}^{l} [\mathbf{Tr}(Q_i S_i + L_i U_i - W_i P_i^0) + \mathbf{Tr}(U_i L_i + R_i T_i)],$$

$$\text{s.t.} \begin{cases} A_i S_i + S_i A_i' + C_i S_i C_i' + B_i U_i + U_i' B_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i T_i D_i' \\ \quad + \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I = 0, \\ \begin{bmatrix} S_i & U_i' \\ U_i & T_i \end{bmatrix} \geqslant 0, \qquad W_i \geqslant 0, \qquad i = 1, \cdots, l, \end{cases} \quad (35)$$

*where $(S_i, T_i, W_i, U_i) \in \mathcal{S}^n \times \mathcal{S}^{n_u} \times \mathcal{S}^n \times \mathbf{R}^{n_u \times n}$ for every $i$.*

THEOREM 4.2. *The dual problem* (35) *is strictly feasible if and only if the system* (8) *is mean-square stabilizable.*

The next theorem is about the existence of the solution of the CGAREs (31) via the SDP (32).

THEOREM 4.3. *The optimal solution set of* (32) *is nonempty and any optimal solution $(P_1^*, \cdots, P_l^*)$ must satisfy the CGAREs* (31).

The following result indicates that any optimal solution of the primal SDP gives rise to a *stabilizing* control of the original LQ problem.

THEOREM 4.4. *Let $(P_1^*, \cdots, P_l^*) \in \mathcal{P}$ be an optimal solution to the primal SDP* (32). *Then the feedback control $u(t) = -\sum_{i=1}^{l} (R_i + D_i' P_i^* D_i)^{-1} (B_i' P_i^* + D_i' P_i^* C_i + L_i') x(t) \chi_{\{r_t = i\}}(t)$ is stabilizing for the system* (8).

THEOREM 4.5. *There exists a unique optimal solution to the SDP* (32), *which is also the maximal solution to the CGAREs* (31).

The theorems in this subsection are proved in the Appendix.

## 4.2. REGULARIZATION

In the previous subsection, we proved our main results under the assumption that the interior of $\mathcal{P}$ was nonempty. Now let us remove this assumption by a regularization argument.

For notational convenience, we rewrite the CGAREs (31) as follows

$$\begin{cases} \mathcal{R}_i(P, Q, R) = 0, \\ R_i + D_i' P_i D_i > 0, \qquad i = 1, \cdots, l, \end{cases} \tag{36}$$

where

$$P = (P_1, \cdots, P_l), \quad Q = (Q_1, \cdots, Q_l), \quad R = (R_1, \cdots, R_l),$$

and

$$\begin{aligned} \mathcal{R}_i(P, Q, R) \triangleq{}& A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j \\ & - (P_i B_i + C_i' P_i D_i + L_i)(R_i + D_i' P_i D_i)^{-1}(B_i' P_i + D_i' P_i C_i + L_i'). \end{aligned}$$

LEMMA 4.1. *Let $Q^1$, $Q^2 \in (\mathcal{S}^n)^l$ and $R^1$, $R^2 \in (\mathcal{S}^{n_u})^l$ be given satisfying $Q_i^1 \leqslant Q_i^2$ and $R_i^1 \leqslant R_i^2$ ($i = 1, \cdots, l$). Assume that there exists $P^0$ such that $\mathcal{R}_i(P^0, Q^1, R^1) > 0$ and $R_i^1 + D_i' P_i^0 D_i > 0$ ($i = 1, \cdots, l$). Then there exist $\overline{P}^1$ and $\overline{P}^2$ satisfying*

$$\begin{cases} \mathcal{R}_i(\overline{P}^1, Q^1, R^1) = 0, \quad R_i^1 + D_i' \overline{P}_i^1 D_i > 0, \\ \mathcal{R}_i(\overline{P}^2, Q^2, R^2) = 0, \quad R_i^2 + D_i' \overline{P}_i^2 D_i > 0, \\ \overline{P}_i^1 \leqslant \overline{P}_i^2, \qquad i = 1, \cdots, l. \end{cases} \tag{37}$$

*Moreover, $\overline{P}^1$ and $\overline{P}^2$ are the maximal solutions of their respective CGAREs.*

*Proof.* By the assumptions, $P^0$ must also satisfy $\mathcal{R}_i(P^0, Q^2, R^2) > 0$ and $R_i^2 + D_i' P_i^0 D_i > 0$ ($i = 1, \cdots, l$). It then follows from Theorem 4.5 that there exist $\overline{P}^1$ and $\overline{P}^2$, which are the maximal solutions of their respective CGAREs:

$$\begin{cases} \mathcal{R}_i(\overline{P}^1, Q^1, R^1) = 0, \quad R_i^1 + D_i' \overline{P}_i^1 D_i > 0, \\ \mathcal{R}_i(\overline{P}^2, Q^2, R^2) = 0, \quad R_i^2 + D_i' \overline{P}_i^2 D_i > 0, \qquad i = 1, \cdots, l. \end{cases} \tag{38}$$

Furthermore, $\overline{P}^1$ must satisfy $\mathcal{R}_i(\overline{P}^1, Q^2, R^2) \geqslant 0$ ($i = 1, \cdots, l$). Hence, for every $i$, $\overline{P}_i^1 \leqslant \overline{P}_i^2$ because $\overline{P}^2$ is the maximal solution to its CGAREs. $\qquad\square$

Let us now present the main result of this section.

THEOREM 4.6. *Let $Q \in (\mathcal{S}^n)^l$ and $R \in (\mathcal{S}^{n_u})^l$ be given. The following are equivalent:*

(i)  *There exists $P^0$ such that $\mathcal{R}_i(P^0, Q, R) \geqslant 0$ and $R_i + D_i' P_i^0 D_i > 0$, $\forall i = 1, \cdots, l$.*

(ii)    *There exists a solution to the CGAREs* (36).
*Moreover, when* (*i*) *or* (*ii*) *holds, the CGAREs* (36) *has a maximal solution* $\overline{P}$
*which is the unique optimal solution to the following SDP problem*

$$\max \ \sum_{i=1}^{l} \mathbf{Tr}(P_i)$$
$$\text{s.t.} \ \begin{cases} \left[ \begin{array}{c|c} A_i'P_i + P_i A_i + C_i'P_i C_i + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j & P_i B_i + C_i'P_i D_i + L_i \\ \hline B_i'P_i + D_i'P_i C_i + L_i' & R_i + D_i'P_i D_i \end{array} \right] \geqslant 0, \\ R_i + D_i'P_i D_i > 0, \qquad\qquad\qquad\qquad i = 1, \cdots, l. \end{cases}$$
$$(39)$$

The proof of this theorem is provided in the Appendix.


## 5.  Optimal LQ Control

In the previous sections we proved that the feasibility of the LMIs is necessary
and sufficient for the solvability of the CGAREs. In this section, we show that the
value function of the LQ problem (8)-(9) can be expressed in terms of the maximal
solution to the CGAREs (36). Moreover, if there exists an optimal control of the LQ
problem then it is necessarily represented as a feedback via the maximal solution
to the CGAREs.

THEOREM 5.1.  *Assume that Theorem* 4.6-(*i*) *holds. Then the LQ problem* (8)-
(9) *is well-posed and the value function is given by* $V(x_0, i) = x_0' \overline{P}_i x_0$, $\forall x_0 \in \mathbf{R}^n$,
$\forall i = 1, 2, \cdots, l$, *where* $\overline{P} = (\overline{P}_1, \cdots, \overline{P}_l)$ *is the maximal solution to the CGAREs*
(36).

THEOREM 5.2.  *Assume that Theorem* 4.6-(*i*) *holds. If there exists an optimal
control of the LQ problem* (8)-(9) *then it must be unique and represented by the
state feedback control* $u(t) = -\sum_{i=1}^{l} (R_i + D_i'\overline{P}_i D_i)^{-1} (B_i'\overline{P}_i + D_i'\overline{P}_i C_i + L_i')x(t)$
$\chi_{\{r_t=i\}}(t)$, *where* $\overline{P} = (\overline{P}_1, \cdots, \overline{P}_l)$ *is the maximal solution to the CGAREs* (36).

Again, the proofs of Theorem 5.1 and Theorem 5.2 can be found in the
Appendix.


## 6.  Numerical Examples

In this section, we report our numerical experiments for a two-mode jump linear
system based on the approach developed in the previous sections. Note that the
numerical algorithm we have used for checking LMIs or solving SDP [11, 18] is
based on an interior-point method [21, 22] which has a polynomial complexity
[20, 21].
    The system dynamics (8) in our experiments is specified by the following
matrices

$$A_1 = \begin{bmatrix} 0.2113249 & 0.3303271 & 0.8497452 \\ 0.7560439 & 0.6653811 & 0.6857310 \\ 0.0002211 & 0.6283918 & 0.8782165 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0683740 & 0.7263507 & 0.2320748 \\ 0.5608486 & 0.1985144 & 0.2312237 \\ 0.6623569 & 0.5442573 & 0.2164633 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.8833888 & 0.9329616 \\ 0.6525135 & 0.2146008 \\ 0.3076091 & 0.3126420 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.3616361 & 0.4826472 \\ 0.2922267 & 0.3321719 \\ 0.5664249 & 0.5935095 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.5015342 & 0.6325745 & 0.0437334 \\ 0.4368588 & 0.4051954 & 0.4818509 \\ 0.2693125 & 0.9184708 & 0.2639556 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.4148104 & 0.7783129 & 0.6856896 \\ 0.2806498 & 0.2119030 & 0.1531217 \\ 0.1280058 & 0.1121355 & 0.6970851 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.8415518 & 0.8784126 \\ 0.4062025 & 0.1138360 \\ 0.4094825 & 0.1998338 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.5618661 & 0.8906225 \\ 0.5896177 & 0.5042213 \\ 0.6853980 & 0.3493615 \end{bmatrix},$$

$$\Pi = \begin{bmatrix} -0.3873779 & 0.3873779 \\ 0.9222899 & -0.9222899 \end{bmatrix}.$$

6.1. NUMERICAL TEST OF MEAN-SQUARE STABILIZABILITY

Consider the LMIs in Lemma 2.5-(vi):

$$\left[ \begin{array}{c|c} A_i X_i + X_i A_i' + B_i Y_i + Y_i' B_i' + \sum_{j=1}^{l} \pi_{ij} X_j & C_i X_i + D_i Y_i \\ \hline X_i C_i' + Y_i' D_i' & -X_i \end{array} \right] < 0, \quad i = 1, 2. \quad (40)$$

We have shown that the controlled system under consideration is mean-square stabilizable if and only if (40) is feasible (with respect to the variables $X_1$, $X_2$, $Y_1$ and $Y_2$). Hence we may check the mean-square stabilizability by solving these LMIs. We find that the following matrices $X_1 = X_1'$, $X_2 = X_2'$, $Y_1 = Y_1'$ and $Y_2 = Y_2'$ satisfy (40):

$$X_1 = \begin{bmatrix} 2140.1643 & 480.89285 & 68.981119 \\ 480.89285 & 848.24038 & -241.62951 \\ 68.981119 & -241.62951 & 198.80034 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 2250.234 & 1049.5331 & 158.56745 \\ 1049.5331 & 1655.5074 & -241.96314 \\ 158.56745 & -241.96314 & 184.02253 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -3040.9847 & -2502.3645 & 389.80811 \\ 883.63967 & 1739.9203 & -831.8653 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 1773.8534 & 1944.5197 & -191.23136 \\ -4966.487 & -4263.3261 & 47.977139 \end{bmatrix},$$

which give rise to the stabilizing feedback control law $u(t) = K_1 x(t)$ (while $r_t = 1$) and $u(t) = K_2 x(t)$ (while $r_t = 2$) with the following feedback gain

$$K_1 = Y_1 X_1^{-1} = \begin{bmatrix} -0.7205287 & -2.9242725 & -1.3434562 \\ 0.2790599 & 1.030096 & -3.0292382 \end{bmatrix},$$

$$K_2 = Y_2 X_2^{-1} = \begin{bmatrix} 0.3719520 & 0.9160973 & -0.1551389 \\ -1.1360428 & -2.0720447 & -1.4848285 \end{bmatrix}.$$

6.2. NUMERICAL SOLUTIONS OF THE CGAREs

Now we proceed to solve the CGAREs (36) for four cases with different $(R_1, R_2)$ under fixed weights $(Q_1, Q_2) = (\mathbf{diag}(1, 0, 1), \mathbf{diag}(1, 1, 0)) \geqslant (0, 0)$ and $(L_1, L_2) = (0, 0)$ via solving the SDP (39).

### (1) $R_1$ and $R_2$ positive definite

Take $R_1 = \mathbf{diag}(1, 1)$ and $R_2 = \mathbf{diag}(1, 2)$. We find the following solution

$$P_1 = \begin{bmatrix} 17.585203 & -12.475468 & -55.006494 \\ -12.475468 & 22.64874 & 61.637189 \\ -55.006494 & 61.637189 & 234.49028 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 13.437397 & 8.2758641 & -9.5632496 \\ 8.2758641 & 22.635405 & 24.101347 \\ -9.5632496 & 24.101347 & 90.563670 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 1.036 \times 10^{-9}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 1.972 \times 10^{-9}$.

### (2) $R_1$ and $R_2$ singular

Take $(R_1, R_2) = (0, 0)$. In this case, we first find that the condition of Theorem 4.6-(i) is satisfied by solving the corresponding LMIs. Hence there must be a maximal solution to the CGAREs (36). We find the following solution

$$P_1 = \begin{bmatrix} 2.8311352 & -2.5169171 & -7.7591546 \\ -2.5169171 & 3.5202908 & 9.7968223 \\ -7.7591546 & 9.7968223 & 32.708219 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.9601908 & 0.1772990 & -1.6708539 \\ 0.1772990 & 2.6087885 & 3.0761238 \\ -1.6708539 & 3.0761238 & 10.969290 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 8.386 \times 10^{-10}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 1.070 \times 10^{-9}$.

### (3) $R_1$ and $R_2$ negative definite

Setting $R_1 = -0.129782I$ and $R_2 = -0.093287I$, we have a maximal solution

$$P_1 = \begin{bmatrix} 1.0736044 & -1.068867 & -1.9944942 \\ -1.068867 & -0.0430021 & 3.0293761 \\ -1.9944942 & 3.0293761 & 8.3215428 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.8369850 & -0.4812074 & -0.5792971 \\ -0.4812074 & -0.1052452 & 0.6058664 \\ -0.5792971 & 0.6058664 & 1.0999906 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 1.310 \times 10^{-11}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 2.740 \times 10^{-11}$.

### (4) $R_1$ and $R_2$ indefinite

Choose

$$R_1 = \begin{bmatrix} -0.01 & 0.03 \\ 0.03 & 0.02 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.01 & 0.03 \\ 0.03 & -0.02 \end{bmatrix}.$$

The negative and positive eigenvalues of $R_1$ are $-0.028541$ and $0.038541$ respectively, and those of $R_2$ are $-0.038541$ and $0.028541$ respectively. We find the following solution

$$P_1 = \begin{bmatrix} 2.496156 & -2.3250539 & -6.7462339 \\ -2.3250539 & 3.1132333 & 8.9556074 \\ -6.7462339 & 8.9556074 & 29.363909 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.7063785 & -0.0240018 & -1.4628917 \\ -0.0240018 & 2.0991262 & 2.6495389 \\ -1.4628917 & 2.6495389 & 9.3117296 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 9.927 \times 10^{-11}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 1.563 \times 10^{-9}$.

## 7. Conclusion

This paper consider a class of stochastic LQ control problems with Markovian jumps in the parameters in infinite time horizon with indefinite state and control cost weighting matrices. The associated CGAREs are extensively investigated, analytically and computationally, via LMIs.

A crucial assumption in the paper is the non-singularity of $R_i + D_i' P_i D_i$ ($i = 1, \cdots, l$). A challenging problem is how to weaken this assumption. Another problem is to extend the LMI technique to the LQ control of jump systems in a finite time horizon where differential Riccati equations have to be involved.

## Appendix

### Proof of Theorem 3.1

(i) For each $(X_1, \cdots, X_l) \in (\mathcal{S}^n)^l$, define the matrices ($i = 1, \cdots, l$):

$$\mathcal{M}_i(X_1, \cdots, X_l)$$
$$\triangleq \left[ \begin{array}{c|c} A_i'X_i + X_iA_i + C_i'X_iC_i + Q_i + \sum_{j=1}^l \pi_{ij}X_j & X_iB_i + C_i'X_iD_i + L_i \\ \hline B_i'X_i + D_i'X_iC_i + L_i' & R_i + D_i'X_iD_i \end{array} \right]. \quad (41)$$

Applying Lemma 2.4, we have

$$\mathcal{P} = \left\{ (P_1, \cdots, P_l) \in (\mathcal{S}^n)^l \,|\, \mathcal{M}_i(P_1, \cdots, P_l) \geqslant 0, \, R_i + D_i'P_iD_i > 0, \, i = 1, \cdots, l \right\}. \quad (42)$$

$\mathscr{P}$ is then seen to be convex as $\mathcal{M}_i(P_1, \cdots, P_l)$, $i = 1, 2, \cdots, l$, are affine in $(P_1, \cdots, P_l)$.

(ii) Take the matrix $\tilde{P}_i$ satisfying (25) in Lemma 3.2 so that $J(x_0, i, u(\cdot)) = x_0'\tilde{P}_i x_0$. For any $(P_1, \cdots, P_l) \in \mathscr{P}$, take $P(r_t) = P_i$ for $r_t = i$, and $u(t) = \sum_{i=1}^{l} K_i x(t) \chi_{\{r_t=i\}}(t)$ with $K_i$ specified in Lemma 3.2. Applying Lemma 3.1, we have

$$E\left\{\int_0^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \,\bigg|\, r_0 = i \right\}$$

$$= x_0' P_i x_0 - E[x(T)'P(r_T)x(T)] + E\left\{\int_0^T x(t)' \mathcal{R}_{r_t}(P_1, \cdots, P_l)x(t) \right. \tag{43}$$

$$\left. + [u(t) - S(r_t)x(t)]'[R(r_t) + D(r_t)'P(r_t)D(r_t)][u(t) - S(r_t)x(t)]dt \,\bigg|\, r_0 = i \right\},$$

where $S(r_t) = S_i \triangleq -[R_i + D_i'P_iD_i]^{-1}[B_i'P_i + D_i'P_iC_i + L_i']$ for $r_t = i$. Letting $T \to +\infty$, we obtain

$$x_0'\tilde{P}_i x_0 = J(x_0, i; u(\cdot)) \geqslant x_0'P_i x_0, \quad \forall x_0 \in \mathbf{R}^n, \quad i = 1, \cdots, l. \tag{44}$$

Since $(P_1, \cdots, P_l) \in \mathscr{P}$ is arbitrary, the desired result follows.  □

## Proof of Theorem 3.2

Fix $(P_1, \cdots, P_l) \in \mathscr{P}$. For any $(x_0, i, u(\cdot)) \in \mathbf{R}^n \times \{1, \cdots, l\} \times \mathcal{U}(x_0, i)$, an easy variant of Lemma 3.1 yields

$$\begin{aligned} &J(x_0, i; u(\cdot)) \\ &= x_0'P_i x_0 + E\left\{\int_0^{+\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \mathcal{M}_{r_t}(P_1, \cdots, P_l) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \,\bigg|\, r_0 = i \right\} \\ &\geqslant x_0'P_i x_0, \end{aligned} \tag{45}$$

due to the fact that $\mathcal{M}_i(P_1, \cdots, P_l) \geqslant 0$ $(i = 1, \cdots, l)$. Hence $V(x_0, i) \geqslant x_0'P_i x_0$. The other statements of the theorem are clear.  □

## Proof of Theorem 4.1

First we show that the constraints of the general dual problem (29), when specializing to the present problem, can be formulated equivalently as the constraints of (35). To this end, define the dual variable $Z = \mathbf{diag}(Z_1, \cdots, Z_l)$ with $Z_i \in \mathcal{S}^{2n+n_u}$ for (29) as

$$Z_i = \left[\begin{array}{cc|c} S_i & U_i' & Y_i' \\ U_i & T_i & \\ \hline Y_i & & W_i \end{array}\right] \geqslant 0,$$

where $(S_i, T_i, W_i, U_i, Y_i) \in \mathcal{S}^n \times \mathcal{S}^{n_u} \times \mathcal{S}^n \times \mathbf{R}^{n_u \times n} \times \mathbf{R}^{n \times (n+n_u)}$. By the general duality relation $\mathbf{Tr}(ZF_i) = c_i, i = 1, \cdots, l$ (see (29)), it follows that for any $(P_1, \cdots, P_l) \in \mathcal{P}$, we have

$$\mathbf{Tr}\{[F(P_1, \cdots, P_l) - F(0, \cdots, 0)]Z\} = -\sum_{i=1}^{l} \mathbf{Tr}(P_i),$$

or equivalently (noting (34)),

$$\sum_{i=1}^{l} \mathbf{Tr}\{[\mathcal{N}_i(P_1, \cdots, P_l) - \mathcal{N}_i(0, \cdots, 0)]Z_i\} = -\sum_{i=1}^{l} \mathbf{Tr}(P_i),$$

which can be written as

$$\sum_{i=1}^{l} \mathbf{Tr}[(A_i S_i + S_i A_i' + C_i S_i C_i' + B_i U_i + U_i' B_i + D_i U_i C_i' + C_i U_i' D_i'$$
$$+ D_i T_i D_i' + \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I)P_i] = 0.$$

This is equivalent to

$$A_i S_i + S_i A_i' + C_i S_i C_i' + B_i U_i + U_i' B_i + D_i U_i C_i' + C_i U_i' D_i' + D_i T_i D_i'$$
$$+ \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I = 0, \qquad i = 1, \cdots, l.$$

On the other hand, the objective of the dual problem (29) reduces to

$$-\sum_{i=1}^{l} \mathbf{Tr}[\mathcal{N}_i(0)Z_i] = -\sum_{i=1}^{l}[\mathbf{Tr}(Q_i S_i + L_i U_i - W_i P_i^0) + \mathbf{Tr}(U_i L_i + R_i T_i)].$$

Notice that the matrix variables $Y_i \in \mathbf{R}^{n \times (n+n_u)}$, $i = 1, 2, \cdots, l$, do not play any role in the above formulation and therefore can be dropped. Finally, the condition $Z_i \geqslant 0$, for every $i$, is equivalent to

$$\begin{bmatrix} S_i & U_i' \\ U_i & T_i \end{bmatrix} \geqslant 0, \qquad W_i \geqslant 0.$$

This completes the proof.                                                      □

## Proof of Theorem 4.2

First assume that the system (8) is mean-square stabilizable by some feedback $u(t) = \sum_{i=1}^{l} K_i x(t) \chi_{\{r_t = i\}}(t)$. Let $\tilde{W}_i > 0$, for every $i$, be fixed. Then by Lemma 2.5-(v) there exists a unique $(S_1, \cdots, S_l)$ satisfying

$$(A_i + B_i K_i)S_i + S_i(A_i + B_i K_i)' + (C_i + D_i K_i)S_i(C_i + D_i K_i)'$$
$$+ \sum_{j=1}^{l} \pi_{ji} S_j + \tilde{W}_i + I = 0, \quad S_i > 0, \qquad i = 1, \cdots, l.$$

Set $U_i = K_i S_i$ $(i = 1, \cdots, l)$. The above relation can then be rewritten as

$$A_i S_i + S_i A_i' + B_i U_i + U_i' B_i' + C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i'$$
$$+ D_i U_i S_i^{-1} U_i' D_i' + \sum_{j=1}^{l} \pi_{ji} S_j + \tilde{W}_i + I = 0, \qquad i = 1, \cdots, l.$$

Let $\epsilon > 0$, and define $T_i = \epsilon I + U_i S_i^{-1} U_i'$ and $W_i = -\epsilon D_i D_i' + \tilde{W}_i$ $(i = 1, \cdots, l)$. Then $T_i$ and $W_i$ satisfy

$$A_i S_i + S_i A_i' + B_i U_i + U_i' B_i' + C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i T_i D_i'$$
$$+ \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I = 0, \qquad i = 1, \cdots, l.$$

Moreover, by Lemma 2.4 for $\epsilon > 0$ sufficiently small we must have

$$\begin{bmatrix} S_i & U_i' \\ U_i & T_i \end{bmatrix} > 0, \qquad W_i > 0, \qquad i = 1, \cdots, l.$$

Therefore, the dual problem (35) is strictly feasible.

Conversely, assume that the dual problem is strictly feasible. Then there exist $S_i > 0$, $T_i$ and $U_i$ $(i = 1, \cdots, l)$ such that

$$\begin{cases} A_i S_i + S_i A_i' + B_i U_i + U_i' B_i' + C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i T_i D_i' \\ \quad + \sum_{j=1}^{l} \pi_{ji} S_j < 0, \\ T_i - U_i S_i^{-1} U_i' > 0, \qquad i = 1, \cdots, l. \end{cases}$$

It follows that

$$A_i S_i + S_i A_i' + B_i U_i + U_i' B_i' + C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i U_i S_i^{-1} U_i' D_i'$$
$$+ \sum_{j=1}^{l} \pi_{ji} S_j < 0, \qquad i = 1, \cdots, l.$$

Define $K_i = U_i S_i^{-1}$ $(i = 1, \cdots, l)$. The above inequality is equivalent to

$$(A_i + B_i K_i) S_i + S_i (A_i + B_i K_i)' + (C_i + D_i K_i) S_i (C_i + D_i K_i)'$$
$$+ \sum_{j=1}^{l} \pi_{ji} S_j < 0, \qquad i = 1, \cdots, l.$$

We conclude that Lemma 2.5-(iii) is satisfied. Hence the system (8) is mean-square stabilizable. □

## Proof of Theorem 4.3

Theorem 4.2, Remark 4.2 along with Proposition 4.1 yield the non-emptiness of the optimal solution set. Next, appealing to the complementary slackness condition (30) in Theorem 4.1, we conclude that any optimal solution $(P_1^*, \cdots, P_l^*)$ must satisfy

$$\begin{bmatrix} A_i' P_i^* + P_i^* A_i + C_i' P_i^* C_i + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j^* & P_i^* B_i + C_i' P_i^* D_i + L_i & 0 \\ B_i' P_i^* + D_i' P_i^* C_i + L_i' & R_i + D_i P_i^* D_i & \\ \hline 0 & & P_i^* - P_i^0 \end{bmatrix}$$
$$\cdot \begin{bmatrix} S_i & U_i' & 0 \\ U_i & T_i & \\ \hline 0 & & W_i \end{bmatrix} = 0,$$

(46)

where $S_i$, $U_i$, $T_i$ and $W_i$ $(i = 1, \cdots, l)$ are the corresponding optimal dual variables. From the above we can deduce the following conditions for $i = 1, 2, \cdots, l$:

$$
\begin{aligned}
&(A_i' P_i^* + P_i^* A_i + C_i' P_i^* C_i + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j^*) S_i \\
&+ (P_i^* B_i + C_i' P_i^* D_i + L_i) U_i = 0,
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&(A_i' P_i^* + P_i^* A_i + C_i' P_i^* C_i + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j^*) U_i' \\
&+ (P_i^* B_i + C_i' P_i^* D_i + L_i) T_i = 0,
\end{aligned}
\tag{48}
$$

$$
(B_i' P_i^* + D_i' P_i^* C_i + L_i') S_i + (R_i + D_i' P_i^* D_i) U_i = 0,
\tag{49}
$$

$$
(B_i' P_i^* + D_i' P_i^* C_i + L_i') U_i' + (R_i + D_i' P_i^* D_i) T_i = 0,
\tag{50}
$$

$$
(P_i^* - P_i^0) W_i = 0.
\tag{51}
$$

Moreover, for every $i$, we have $R_i + D_i' P_i^* D_i > 0$, since $R_i + D_i' P_i^0 D_i > 0$. Hence, (49) implies that $U_i = -(R_i + D_i' P_i^* D_i)^{-1}(B_i' P_i^* + D_i' P_i^* C_i + L_i') S_i$ $(i = 1, \cdots, l)$. Putting this into equation (47) leads to $\mathcal{R}_i(P_1^*, \cdots, P_l^*) S_i = 0$. A same manipulation of equations (48) and (50) yields $\mathcal{R}_i(P_1^*, \cdots, P_l^*) U_i' = 0$. Recall that the dual variables $S_i$, $U_i$, $T_i$ and $W_i$, for every $i$, satisfy the following constraint

$$
\begin{aligned}
&A_i S_i + S_i A_i' + B_i U_i + U_i' B_i' + C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' \\
&+ D_i T_i D_i' + \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I = 0.
\end{aligned}
\tag{52}
$$

Multiplying both sides of the above by $\mathcal{R}_i(P_1^*, \cdots, P_l^*)$ we have

$$
\begin{aligned}
&\mathcal{R}_i(P_1^*, \cdots, P_l^*)[C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i T_i D_i' \\
&+ \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I] \mathcal{R}_i(P_1^*, \cdots, P_l^*) = 0.
\end{aligned}
$$

Since

$$
\begin{bmatrix} S_i & U_i' \\ U_i & T_i \end{bmatrix} \geqslant 0, \qquad i = 1, \cdots, l,
\tag{53}
$$

it follows from Lemma 2.4 that $T_i \geqslant U_i S_i^\dagger U_i'$ and $U_i = U_i S_i S_i^\dagger$. These imply, for every $i$,

$$
\begin{aligned}
&\mathcal{R}_i(P_1^*, \cdots, P_l^*)[C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i U_i S_i^\dagger U_i' D_i' \\
&+ \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I] \mathcal{R}_i(P_1^*, \cdots, P_l^*) \leqslant 0.
\end{aligned}
\tag{54}
$$

By virtue of Lemma 2.3, we deduce the following

$$
\begin{aligned}
&C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i U_i S_i^\dagger U_i' D_i' \\
&= C_i S_i S_i^\dagger S_i C_i' + D_i U_i S_i^\dagger S_i C_i' + C_i S_i S_i^\dagger U_i' D_i' + D_i U_i S_i^\dagger U_i' D_i' \\
&= (C_i S_i + D_i U_i) S_i^\dagger (S_i C_i' + U_i' D_i') \geqslant 0.
\end{aligned}
\tag{55}
$$

Then it follows from (54) that $\mathcal{R}_i(P_1^*, \cdots, P_l^*)\mathcal{R}_i(P_1^*, \cdots, P_l^*) \leqslant 0$, resulting in $\mathcal{R}_i(P_1^*, \cdots, P_l^*) = 0$ $(i = 1, \cdots, l)$. $\square$

## Proof of Theorem 4.4

Let $S_i$, $T_i$, $U_i$ and $W_i$ $(i = 1, \cdots, l)$ be the corresponding optimal dual variables satisfying (47)–(51). First, we are to show that $S_i > 0$ $(i = 1, \cdots, l)$. Suppose that $S_i x = 0$, $x \in \mathbf{R}^n$, for a fixed $i$. As $U_i$ satisfies

$$U_i = -(R_i + D_i' P_i^* D_i)^{-1}(B_i' P_i^* + D_i' P_i^* C_i + L_i')S_i \qquad (56)$$

(see (49)), we also have $U_i x = 0$. The dual constraint (52) then implies

$$x'[C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i U_i S_i^{\dagger} U_i' D_i' + \sum_{j=1}^{l} \pi_{ji} S_j + W_i + I]x \leqslant 0.$$

The same manipulation as in the proof of Theorem 4.3 gives $x = 0$. As $S_i \geqslant 0$, we conclude that $S_i > 0$. Now, the equality (52) gives

$$\begin{cases} A_i S_i + S_i A_i' + B_i U_i + U_i' B_i' + C_i S_i C_i' + D_i U_i C_i' + C_i U_i' D_i' + D_i U_i S_i^{-1} U_i' D_i' \\ \quad + \sum_{j=1}^{l} \pi_{ji} S_j < 0, \\ S_i > 0, \qquad\qquad\qquad\qquad\qquad\qquad i = 1, \cdots, l, \end{cases}$$

which is equivalent to the mean-square stabilizability condition given by Lemma 2.5-(iii) with $K_i = -(R_i + D_i' P_i^* D_i)^{-1}(B_i' P_i^* + D_i' P_i^* C_i + L_i')$. $\square$

## Proof of Theorem 4.5

Let $(P_1^*, \cdots, P_l^*) \in \mathscr{P}$ be an optimal solution to the SDP (32). Theorem 4.3 shows that $(P_1^*, \cdots, P_l^*)$ solves the CGAREs (31). To show that it is indeed a maximal solution, define $K_i = -(R_i + D_i' P_i^* D_i)^{-1}(B_i' P_i^* + D_i' P_i^* C_i + L_i')$. A simple calculation yields

$$(A_i + B_i K_i)' P_i^* + P_i^*(A_i + B_i K_i) + (C_i + D_i K_i)' P_i^*(C_i + D_i K_i)$$
$$+ \sum_{j=1}^{l} \pi_{ij} P_j^* = -Q_i - L_i K_i - K_i' L_i' - K_i' R_i K_i.$$

On the other hand, it follows from Theorem 4.4 that $u^*(t) = -\sum_{i=1}^{l}(R_i + D_i' P_i^* D_i)^{-1}(B_i' P_i^* + D_i' P_i^* C_i + L_i')x(t)\chi_{\{r_t=i\}}(t)$ is a stabilizing control. A proof similar to that of Theorem 3.1-(ii) yields that $(P_1^*, \cdots, P_l^*)$ is the upper bound of the set $\mathscr{P}$, namely, $(P_1^*, \cdots, P_l^*)$ is the maximal solution. In addition, the uniqueness of the solution to the SDP (32) follows from the maximality. This completes the proof. $\square$

**Proof of Theorem 4.6**

We only need to prove that (i) implies (ii). Let $P^0$ be given as in (i). For any $\epsilon > 0$ we have $\mathcal{R}_i(P^0, Q + \epsilon\mathbf{I}, R) > 0$ $(i = 1, \cdots, l)$. Applying Theorem 4.5 and Lemma 4.1, we have that for any positive decreasing sequence $\epsilon_k \to 0$ there exists a decreasing sequence of symmetric matrices

$$P_i^{\epsilon_0} \geqslant \cdots \geqslant P_i^{\epsilon_k} \geqslant P_i^{\epsilon_{k+1}} \geqslant P_i^0, \qquad i = 1, \cdots, l$$

such that $\mathcal{R}_i(P^{\epsilon_k}, Q + \epsilon_k\mathbf{I}, R) = 0$ and $R_i + D_i' P_i^{\epsilon_k} D_i > 0$. Hence, for every $i$, the limit $\overline{P}_i = \lim\limits_{\epsilon_k \to 0} P_i^{\epsilon_k}$ exists and satisfies $\mathcal{R}_i(\overline{P}, Q, R) = 0$, $R_i + D_i'\overline{P}_i D_i > 0$. In addition, $\overline{P}$ must be the maximal solution of the CGAREs due to the arbitrariness of $P^0$. To show the uniqueness, let $(\widetilde{P}_1, \cdots, \widetilde{P}_l) \in \mathcal{P}$ be any optimal solution to (39). Hence, $\sum_{i=1}^l \mathbf{Tr}(\overline{P}_i - \widetilde{P}_i) = 0$. However, $\overline{P}_i - \widetilde{P}_i \geqslant 0$, $\forall i = 1, \cdots, l$, since $(\overline{P}_1, \cdots, \overline{P}_l)$ is the maximal solution of (36). This leads to $\overline{P}_i - \widetilde{P}_i = 0$, $\forall i = 1, \cdots, l$, which completes the proof. $\qquad\square$

**Proof of Theorem 5.1**

The well-posedness has been shown in Theorem 3.2, which also yields $V(x_0, i) \geqslant x_0'\overline{P}_i x_0$.

Now, for any fixed $\epsilon > 0$, the LMIs

$$\mathcal{R}_i(P, Q + \epsilon\mathbf{I}, R) \geqslant 0, \quad R_i + D_i' P_i D_i > 0, \quad i = 1, 2, \cdots, l \qquad (57)$$

are strictly feasible. Hence by Theorem 4.5, there is a maximal solution, denoted by $P^\epsilon = (P_1^\epsilon, \cdots, P_l^\epsilon)$, to the corresponding CGAREs

$$\mathcal{R}_i(P, Q + \epsilon\mathbf{I}, R) = 0, \quad R_i + D_i' P_i D_i > 0, \qquad i = 1, \cdots, l.$$

In addition, by Theorem 4.4, the feedback control $u^\epsilon(t) = \sum_{i=1}^l K_i^\epsilon x^\epsilon(t)\chi_{\{r_t=i\}}(t)$ is stabilizing, where $K_i^\epsilon = -(R_i + D_i' P_i^\epsilon D_i)^{-1}(B_i' P_i^\epsilon + D_i' P_i^\epsilon C_i + L_i')$. It is easy to verify that $P^\epsilon$ and $(K_1^\epsilon, \cdots, K_l^\epsilon)$ satisfy the following coupled Lyapunov equations

$$\begin{aligned}
&(A_i + B_i K_i^\epsilon)' P_i^\epsilon + P_i^\epsilon(A_i + B_i K_i^\epsilon) + (C_i + D_i K_i^\epsilon)' P_i^\epsilon(C_i + D_i K_i^\epsilon) \\
&+ \sum_{j=1}^l \pi_{ij} P_j^\epsilon = -Q_i - L_i K_i^\epsilon - K_i^{\epsilon\prime} L_i' - K_i^{\epsilon\prime} R_i K_i^\epsilon, \qquad i = 1, \cdots, l.
\end{aligned} \tag{58}$$

Applying Lemma 3.2 to (58), we have

$$x_0' P_i^\epsilon x_0 = E\left\{\int_0^{+\infty} \begin{bmatrix} x^\epsilon(t) \\ u^\epsilon(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) + \epsilon I & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x^\epsilon(t) \\ u^\epsilon(t) \end{bmatrix} dt \,\bigg|\, r_0 = i \right\} \geqslant V(x_0, i).$$

On the other hand, since $\overline{P}_i = \lim\limits_{\epsilon \to 0} P_i^\epsilon$ (as in the proof of Theorem 4.6) we have $V(x_0, i) \leqslant x_0'\overline{P}_i x_0$. This concludes the proof. $\qquad\square$

**Proof of Theorem 5.2**

Define $K(r_t) = K_i \triangleq -(R_i + D_i' \overline{P}_i D_i)^{-1}(B_i' \overline{P}_i + D_i' \overline{P}_i C_i + L_i')$ whenever $r_t = i$. Let $(\overline{x}(\cdot), \overline{u}(\cdot))$ be an optimal pair of the LQ problem. Then a completion of squares shows

$$
\begin{aligned}
& E\left\{\int_0^T \begin{bmatrix} \overline{x}(t) \\ \overline{u}(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \overline{u}(t) \end{bmatrix} dt \,\Big|\, r_0 = i \right\} \\
& = x_0' \overline{P}_i x_0 - E[\overline{x}(T)' \overline{P}(r_T) \overline{x}(T)] \\
& \quad + E \int_0^T [\overline{u}(t) - K(r_t)\overline{x}(t)]' [R(r_t) + D(r_t)' \overline{P}(r_t) D(r_t)]^{-1} \\
& \quad \cdot [\overline{u}(t) - K(r_t)\overline{x}(t)] dt.
\end{aligned}
$$

As $\overline{u}(\cdot)$ is stabilizing, $\lim\limits_{T \to +\infty} E[\overline{x}(T)' \overline{P}(r_T) \overline{x}(T)] = 0$, which implies

$$
\begin{aligned}
& V(x_0, i) \\
& = J(x_0, i; \overline{u}(\cdot)) \\
& = x_0' \overline{P}_i x_0 + E \int_0^{+\infty} [\overline{u}(t) - K(r_t)\overline{x}(t)]' [R(r_t) + D(r_t)' \overline{P}(r_t) D(r_t)]^{-1} \\
& \quad \cdot [\overline{u}(t) - K(r_t)\overline{x}(t)] dt.
\end{aligned} \tag{59}
$$

By Theorem 5.1, we have $V(x_0, i) = x_0' \overline{P}_i x_0$. Hence,

$$
E \int_0^{+\infty} [\overline{u}(t) - K(r_t)\overline{x}(t)]' [R(r_t) + D(r_t)' \overline{P}(r_t) D(r_t)]^{-1} [\overline{u}(t) - K(r_t)\overline{x}(t)] dt = 0. \tag{60}
$$

As, for every $i$, $R_i + D_i' \overline{P}_i D_i$ is a constant positive definite matrix, $\overline{u}(t)$ has to be in a feedback form $\overline{u}(t) = K(r_t)\overline{x}(t) = \sum_{i=1}^l K_i \overline{x}(t) \chi_{\{r_t = i\}}(t)$. This completes the proof. $\qquad\square$

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